1. Let R_1 be the relation on $\mathbb{Z}$ defined by aR_1b if and only if $ab \geq 0$ and let R_2 be the relation on $\mathbb{Z} \setminus \{0\}$ defined by aR_2b if and only if $ab > 0$, (where $\mathbb{Z}$ is the set of integers)

   (a) Test whether R_1 is an equivalence relation on $\mathbb{Z}$. Justify your answer.

   (b) Test whether R_2 is an equivalence relation on $\mathbb{Z} \setminus \{0\}$. Justify your answer. (5 Marks)

2. Is it true that there exists a bijective function from $\mathbb{N}$ *to* $\mathbb{Z}$? If yes, define such a function explicitly and prove that it is bijective. Justify your reasoning with clear steps, (where $\mathbb{N}$ is the set of natural numbers). (5 Marks)

3. Using the Euclidean algorithm, find the greatest common divisor (gcd) of 252 and 198. Then Express this gcd as a linear combination of 252 and 198. (5 Marks)

4. Using Fermat's Little Theorem, find the remainder when 5^{101} is divided by 17. (5 Marks)

5. Solve the reciprocal equation $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$ by using the reciprocal transformation. (5 Marks)

6. Let $f(x) = x^4 - 5x^2 + 6$. Let the roots be $\alpha, \beta, \gamma, \delta$. Compute $\sum \alpha^2$, and also find $\sum \alpha^2 \beta^2$? (5 Marks)

7. Let $\mathbb{R}$ be the group of real numbers under addition. Determine to which of the following groups it is isomorphic. Justify your answer with proof.

   (a) The group $\mathbb{R}^+$ of positive real numbers under multiplication.

   (b) The group $\mathbb{C}^*$ of non-zero complex numbers under multiplication. (5 Marks)

8. Find all ideals N of the ring $\mathbb{Z}_6$. For each ideal N, compute the quotient ring $\mathbb{Z}_6/N$ and identify a known ring (upto isomorphism) to which $\mathbb{Z}_6/N$ is isomorphic. Justify your answers. (5 Marks)

9. Find a basis for the vector space $M_{2\ 2}$, the space of all $2 \quad 2$ matrices with real entries. (5 Marks)

10. Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(1, 0, 0) = \left(1, -\frac{3}{2}, 2\right)$$

$$T(0,1,0) = \left(-3, \frac{9}{2}, -6\right)$$

$$T(0,0,1) = (2, -3, 4)$$

Find nullity of T. (5 Marks)

11. Find the rank of the following matrix. (5 Marks)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \end{pmatrix}$$

12. Let V be the vector space of all complex polynomials p with $deg(p) \leq n$. Let $T: V \to V$ be the map $(T_p)(x) = p'(1)$. Prove that $dim(ker(T)) = n$. (5 Marks)

13. For any nonempty bounded set $A \subset \mathbb{R}$ define $-A = \{-a : a \in A\}$. Prove that $in\ f(-A) = -sup(A)$. (5 Marks)

14. Let $0 < b < 1$. Is the sequence (na^n) convergent? Justify your answer. (5 Marks)

15. Discuss the convergence of the series $\sum_{n=1}^{\infty} n!\, e^{-n^2}$ (5 Marks)

16. Prove that if S and T are countable, then $S \cup T$ is countable. (5 Marks)

17. Let f: $\mathrm{R} \to \mathrm{R}$ be differentiable and $\lim_{x \to \infty} f'(x) = 1$. Show that f is unbounded. (5 Marks)

18. State TRUE or FALSE by giving a proper justification : There exists a continuous function $f: R \to R$ such that $f(x) \in Q$ for all $x \in R - Q$ and $f(x) \in R - Q$ for all $x \in Q$. (5 Marks)

19. Let f: $\mathrm{R} \to \mathrm{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. If f is continuous at 0, then show that, $f(x) = f(1)x$ for all $x \in R$. (5 Marks)

20. Let p be an odd degree polynomial with real coefficients in one real variable. If $g: R \to R$ is a bounded continuous function, then show that there exists $x_0 \in R$ such that $p(x_0) = g(x_0)$. Apply this result to prove that the equation $x^9 - 4x^6 + x^5 + \dfrac{1}{1+x^2} = \sin 3x + 17$ has atleast one real root.

(5 Marks)

21. Show that the Dirichlet function

$$f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$$

is not integrable on [0, 1]. (5 Marks)

22. Find area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.
(5 Marks)

23. Find the length of the loop of the curve $3ay^2 = x(x-a)^2$. (5 Marks)

24. Use the method of slicing, find the volume a right circular cone whose height is h and base is a circle of radius r. (5 Marks)

25. Show that the locus of foot of the perpendicular from the origin to plane passing through a fixed point passes through the origin. (5 Marks)

26. Determine the point at which the plane $2x + 2y + z - 8 = 0$ is tangential to the sphere $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$. (5 Marks)

27. Find the line of intersection of the planes $2x + 3y + 4z - 5 = 0$ and $x + 2y + 3z - 4 = 0$. (5 Marks)

28. Using the geometrical interpretation of scalar triple product of vectors, verify whether the points $(1,-1,-5),(4, 2, 1),(5, 3, -3)$ and $(-2, -2, -9)$ are coplanar. (5 Marks)

29. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 6$ and $z = 0$. (5 Marks)

30. Use a polar double integral to find the area enclosed by the cardioid $r = 1 + \cos\theta$. (5 Marks)

31. Use Green's Theorem to find the work done by the force field $\vec{F}(x, y) = (x^2 + y)\vec{i} + (y^2 - x)\vec{j}$ on a particle that travels once counter clockwise around the triangle bounded by $x = 0,\ y = 0,\ x + y = 2$. (5 Marks)

32. Use the Divergence Theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = 4x\vec{i} + y\vec{j} + z^2\vec{k}$ across the unit cube given by the inequalities $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$. (5 Marks)

33. Find the orthogonal trajectories of family of curves $y = x + ce^{-x}$. (5 Marks)

34. Find the integrating factor of the equation and then solve the equation $ydx + (2x - ye^{y})\,dy = 0$. (5 Marks)

35. (a) Define (i) Slack variable (ii) unrestricted variable

(b) Change the following LPP into standard form

Maximize $Z = 2x_1 + 2x_2 + 4x_3 + x_4$

Subject to the constraints

$3x_1 + 2x_2 + x_3 + 9x_4 \le 16$

$7x_1 + 2x_2 + 4x_3 + x_4 \le 24$

$x_1, x_2, x_3 \ge 0$ and x_4 unrestricted (5 Marks)

36. Obtain an initial basic feasible solution using north west corner rule.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

(5 Marks)

37. Prove that the function $U(x, y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$ is harmonic. Find an analytic function $f(z)$ so that the real part of $f(z)$ is $U(x, y)$. (5 Marks)

38. Find the maximum value of $|f(z)|$ on the rectangle $-1 \leq \text{Re}(z) \leq 1$, $1 \leq Im(z) \leq 2$, where $f(z) = \dfrac{e^z}{z}$. (5 Marks)

39. Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ as a power series in the region :

(a) $|z| < 1$

(b) $1 < |z| < 2$

(c) $|z| > 2$ (5 Marks)

40. Using Cauchy's Residue theorem, evaluate $\oint_C \left(\dfrac{ze^{\pi z}}{z^4 - 16} + z^n e^{\frac{\pi}{z}} \right) dz$, where n is an integer with $n \geq 1$ and C is the ellipse $9x^2 + y^2 = 9$. (20 Marks)
